An Introduction to Mapping Class Groups

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Abstract

We provide an introduction to the mapping class group, which is the group of the symmetries of a surface. Firstly, we describe basic properties of surfaces. Next, we consider homeomorphisms of surfaces that preserve its topological properties. A special example of such is a Dehn twist. We define a Dehn twist on an annulus and then extend the notion to a Dehn twist about a simple closed curve on a surface. As a result, we find that Dehn twists are important elements of the mapping class group. In fact, they generate the mapping class group of a compact orientable surface. Lastly, we give examples of trivial mapping class groups and explore mapping class groups of infinite order, namely that of the annulus and torus. We conclude with additional remarks about braid groups and their connection to mapping class groups.

Acknowledgements

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1 Introduction

Symmetries are ubiquitous in the natural world and one can learn about an object by studying its group of symmetries. This is a common theme in mathematics; for example, in abstract algebra, the study of the symmetric group and the dihedral group uncovers the beauty of two fundamental objects, namely, a finite set and a regular polygon, respectively. In this expository piece, we delve into the realm of surfaces to explore this fundamental object in topology. We dedicate the entire next section to describe basic properties and homeomorphisms of surfaces. We also define being homotopic as an equivalence relation, as it is an important notion to consider when handling simple closed curves on surfaces. Then we highlight a special example of a homeomorphism called a Dehn twist. Since we consider orientable surfaces, we have that annulus neighborhoods exist, and, thus, we can apply a Dehn twist to obtain homotopic curves and homeomorphisms that belong to the same mapping class. Together, these mapping classes form a group, which leads to a formal definition of a mapping class group of a surface. Next, we give examples of trivial mapping class groups, such as that of S^1, S^2 and D^2. Lastly, we look at the annulus and torus to prove interesting results about these mapping class groups of infinite order.

Geometric group theory is devoted to the study of groups and the spaces on which they act. As important algebraic invariants of topological spaces, mapping class groups play an significant role in the study of 3-dimensional manifolds. They are also valuable in understanding the topology of surfaces. In fact, their origins lie in the topology of hyperbolic surfaces, especially, the study of the intersections of closed curves on hyperbolic surfaces. Since then, we have discovered a plethora of remarkable attributes of the mapping class group of surfaces. It is related to various other groups, including braid groups.

2 Surfaces

Definition 2.1 (surface)**.** A surface is a two-dimensional manifold with or without boundary. Informally, it is a geometrical shape that resembles a deformed plane.

Example 2.1. Boundaries of solid objects in $\mathbb{R}^3$, such as a sphere and torus, are the most familiar examples.

Figure 1: A list of surfaces without boundary.

2.1 Basic properties

Definition 2.2 (compact)**.** A compact surface is a surface that is also a closed and bounded set.

Definition 2.3 (surface with boundary)**.** Let S be a surface. The boundary of S is the collection of points on S minus the set of all interior points of S, i.e. $\partial S = S \setminus \mathrm{int}(S)$. If $\partial S \neq \emptyset$, then S is a surface with boundary.

Example 2.2. Some examples of surfaces with boundary include a disk, annulus, pair of pants, and torus with a disk removed.

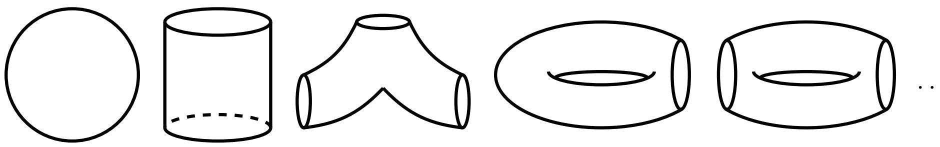

Figure 2: A list of surfaces with boundary.

Remark 2.1. An *orientable* surface allows a consistent definition of "clockwise" and "counterclockwise." On the other hand, a surface is *non-orientable* if and only if it contains a Möbius band.

Remark 2.2. The *genus* g of an orientable surface S is an integer representing the number of handles, or holes, on S.

Example 2.3. In Figure 1, the surfaces are listed in ascending order from genus 0 to genus n.

Definition 2.4 (homeomorphism)**.** Let S be a surface. A homeomorphism $f : S \to S$ is a continuous bijection with a continuous inverse.

Remark 2.3. Some examples include rotations, reflections and hyperelliptic involutions. Note that the orientation changes under a reflection. Moreover, a special example of a homeomorphism that cannot be realized by rigid motions is a Dehn twist. We dedicate the entire next section to this topic.

Example 2.4. Using polar coordinates, define the rotation by angle θ as

$$\begin{aligned} f_\theta &: S^1 \to S^1 \\ &(1, \alpha) \mapsto (1, \alpha + \theta), \end{aligned}$$

where α is any angle.

Theorem 2.1 (Classification of Surfaces)**.** Every compact, orientable surface without boundary is homeomorphic to one of the surfaces from Figure 1. See [3] for more details.

Remark 2.4. In other words, compact, orientable surfaces without boundary are homeomorphic if and only if they share the same genus g.

2.2 Homotopy between homeomorphisms

Definition 2.5 (homotopic)**.** Let S be an orientable surface and let $f : S \to S$ and $g : S \to S$ be two homeomorphisms. We call f and g homotopic if there exists a continuous map $H : S \times [0,1] \to S$ such that $H_0 = f$ and $H_1 = g$, where $H_t(x) = H(x,t)$.

Proposition 2.1. The identity map id_{S^1} is homotopic to the homeomorphism f_θ from Example 2.4.

Proof. Consider the map

$$\begin{aligned} H : S^1 \times [0,1] &\to S^1 \\ (\alpha, 0) &\mapsto \alpha \\ (\alpha, 1) &\mapsto \alpha + \theta. \end{aligned}$$

Then $H(\alpha,t) = \alpha + \theta t$, where $H_0 = id_{S^1}$ and $H_1 = f_\theta$. Thus, id_{S^1} is homotopic to f_θ. $\square$

Proposition 2.2. The identity map id_{S^2} is homotopic to the homeomorphism $f_\theta : S^2 \to S^2$.

Proof. Choose an axis of rotation, say the z-axis, and define the rotation map by angle θ of S^2

$$\begin{aligned} f_\theta : S^2 &\to S^2 \\ (r, \alpha, z_0) &\mapsto (r, \alpha + \theta, z_0 + z), \end{aligned}$$

where radius $r > 0$, angle $\alpha \in [-\pi, \pi]$ and $z_0, z \in \mathbb{R}$ is the usual z-coordinate in the Cartesian coordinate system. Then using a similar idea as in Proposition 2.1, consider

$$\begin{aligned} H : S^2 \times [0,1] &\to S^2 \\ (\alpha, z_0, 0) &\mapsto (\alpha, z_0) \\ (\alpha, z_0, 1) &\mapsto (\alpha + \theta, z_0 + z). \end{aligned}$$

We omit writing r for convenience. It follows that $H(\alpha, z_0, t) = (\alpha + \theta t, z_0 + zt)$, where $H_0 = id_{S^2}$ and $H_1 = f_\theta$. So, id_{S^2} is indeed homotopic to $f_\theta : S^2 \to S^2$. $\square$

Lemma 2.1. Homotopy defines an equivalence relation.

Proof. We need to show that homotopy satisfies reflexivity, symmetry and transitivity.

$\boxed{\text{Reflexive}}$ Let $f : X \to Y$ be a continuous function and define $H : X \times [0,1] \to Y$ such that $H_0 = H_1 = f$. Then clearly $H(x,t) = f(x)$. Thus, f is homotopic to itself.

$\boxed{\text{Symmetric}}$ Suppose f is homotopic to g. Then there exists a homotopy $F : X \times [0,1] \to Y$ such that $F_0 = f$ and $F_1 = g$. We want to show that there exists a homotopy $H : X \times [0,1] \to Y$ such that $H_0 = g$ and $H_1 = f$. Since $H(x,0) = F(x,1)$ and $H(x,1) = F(x,0)$, we can set $H(x, \frac{1}{2}) = F(x, \frac{1}{2})$. Therefore, we can choose $H(x,t) = F(x, 1-t)$, so g is homotopic to f.

$\boxed{\text{Transitive}}$ Lastly, suppose f is homotopic to g and g is homotopic to h. Given $F : X \times [0,1] \to Y$ such that $F_0 = f$ and $F_1 = g$ and $G : X \times [0,1] \to Y$ such that $G_0 = g$ and $G_1 = h$, we want to construct a homotopy $H : X \times [0,1] \to Y$ such that $H_0 = f$ and $H_1 = h$. Notice that $H(x,0) = F(x,0)$ and $H(x,1) = G(x,1)$. Moreover, $F(x,1) = G(x,0)$. Take $H(x, \frac{1}{2}) = F(x,1) = G(x,0)$. Then we can write

$$H(x,t) = \begin{cases} F(x,2t) & \text{if } t \in [0, \frac{1}{2}] \\ G(x,2t) & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

Hence, our proof is complete. $\square$

3 Dehn twists

We return to a special example of a homeomorphism, namely a Dehn twist. First, we look at a Dehn twist on an annulus and then on a surface. Using polar coordinates (r, θ) for points in the plane $\mathbb{R}^2$, we consider the annulus A made up of those points with $1 \leq r \leq 2$.

Definition 3.1 (Dehn twist on A)**.** We define a Dehn twist on A as follows:

$$\begin{aligned} T_A : A &\longrightarrow A \\ (r, \theta) &\longmapsto (r, \theta - 2\pi r), \end{aligned}$$

where the boundary of A, denoted ∂A, is fixed pointwise.

Remark 3.1. We can consider the *core* of an annulus A, which is the set of points when $r = \frac{3}{2}$. Every simple closed curve, i.e. loops without self-intersections, on an orientable surface is the core of some annulus.

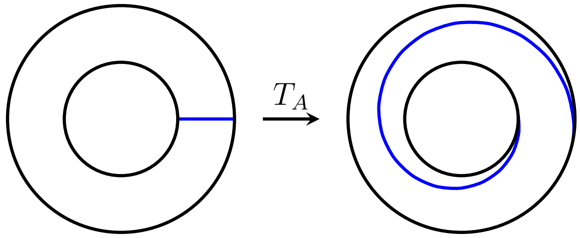

Figure 3: A Dehn twist on an annulus A.

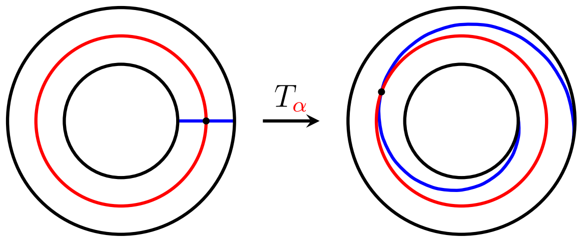

Figure 4: A Dehn twist about the red simple closed curve α.

Definition 3.2 (Dehn twist on a surface)**.** Let S be an orientable surface with two simple closed curves α and β. Then a Dehn twist about α on S is obtained by choosing an annulus A, applying T_α and extending by the identity, i.e. fixing every point in $S \setminus A$. Similarly, a Dehn twist about β on S is obtained by choosing an annulus A, applying T_β and extending by the identity.

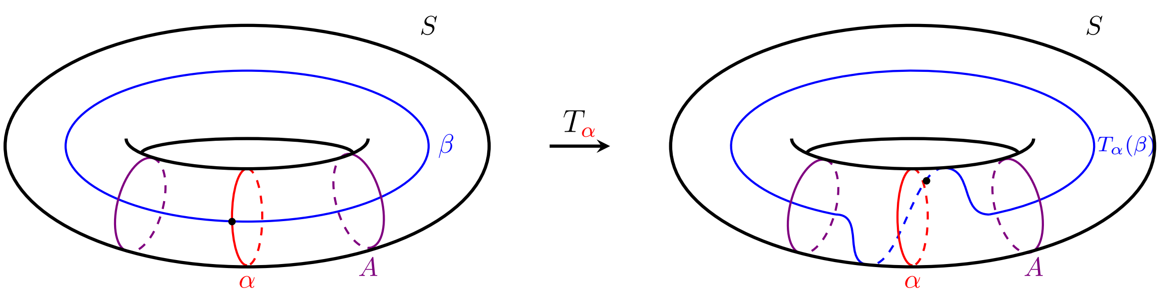

Figure 5: We realize the simple closed curve α as the core of annulus A.

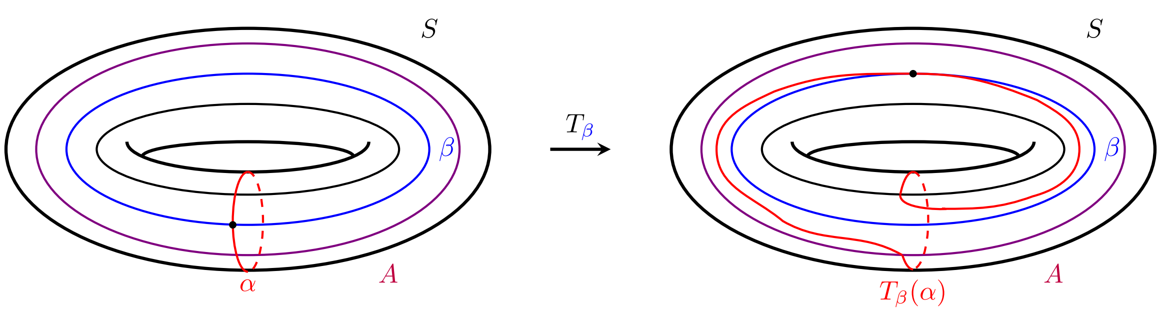

Figure 6: We realize the simple closed curve β as the core of annulus A.

Remark 3.2. In Figure 5, suppose we choose a different annulus A'. Then we will obtain a curve $T'_\alpha(\beta)$ that is homotopic to $T_\alpha(\beta)$. Moreover, we have that T_α and T'_α are homotopic homeomorphisms and, thus, belong to the same mapping class, which we will define in the next section. In particular, any choice of annulus will yield an element of the mapping class of T_α.

4 Mapping Class Groups

We can consider the set of all homeomorphisms of a surface S, denoted by Homeo(S). For surfaces with boundary, we only consider the homeomorphisms that fix ∂S pointwise.

Remark 4.1. The set of all homeomorphisms that are homotopic to identity 1_S is denoted by $\text{Homeo}_0(S)$.

Lemma 4.1. Homeo(S) is a group, with $\text{Homeo}_0(S)$ as a normal subgroup.

Proof. Clearly, Homeo(S) is a group since function composition is associative, $1_S \in \text{Homeo}(S)$ and for any $f \in \text{Homeo}(S)$, its inverse $f^{-1} \in \text{Homeo}(S)$, by definition. Moreover, $\text{Homeo}_0(S)$ is a subgroup of Homeo(S). Note that $1_S \in \text{Homeo}_0(S)$ since 1_S is homotopic to itself. Let $f, g \in \text{Homeo}_0(S)$. We can define the homotopy $H(x,t) = F(G(x,t),t)$, where $F : S \times [0,1] \to S$ is a homotopy such that $F_0 = 1_S$ and $F_1 = f$ and $G : S \times [0,1] \to S$ is a homotopy such that $G_0 = 1_S$ and $G_1 = g$ for all $x \in S$. Then $H_0(x) = x$ and $H_1(x) = f(g(x))$. So it follows that $\text{Homeo}_0(S)$ is closed under function composition. Furthermore, let $f, g, h \in \text{Homeo}(S)$ with f and g homotopic. Then $h \circ f$ is homotopic to $h \circ g$ since we can construct the homotopy $K(x,t) = h(H(x,t))$ such that $H : S \times [0,1]$, where $H_0 = f$ and $H_1 = g$. Using this fact, we deduce that for any $f \in \text{Homeo}_0(S)$, also $f^{-1} \in \text{Homeo}_0(S)$ when we notice that $f^{-1} \circ f$ is homotopic to $f^{-1} \circ 1_S$.

Now we show that $\text{Homeo}_0(S) \trianglelefteq \text{Homeo}(S)$. It suffices to show that for any $g \in \text{Homeo}(S)$ and $f \in \text{Homeo}_0(S)$, we have that $gfg^{-1} \in \text{Homeo}_0(S)$. Since $f \in \text{Homeo}_0(S)$, there is $F : S \times [0,1] \to$ such that $F_0 = f$ and $F_1 = 1_S$. Note that for any $0 \le t \le 1$,

$$H(x,t) = g(F(g^{-1}(x),t))$$

such that $H_0 = gfg^{-1}$ and $H_1 = 1_S$ is continuous. Therefore, our proof is complete. $\square$

Definition 4.1 (Fundamental group)**.** The fundamental group of a topological space X, denoted $\pi_1(X,x)$, is the group of homotopy classes of x-based loops in X.

Remark 4.2. In Section 3, we deduced that if f is homotopic to g and α is a simple closed curve, then $f(\alpha)$ and $g(\alpha)$ are homotopic curves. The fundamental group of a surface captures the group structure of equivalence classes of simple closed curves on a surface.

4.1 Definitions and elementary examples

Now we can introduce a formal definition of the mapping class group.

Definition 4.2 (Mapping class group)**.** Let S be an orientable surface. The mapping class group of S, denoted by MCG(S), is the group of homotopy classes of orientation-preserving homeomorphisms of S, i.e. $\mathrm{MCG}(S) = \mathrm{Homeo}_+(S) \ / \ \mathrm{Homeo}_0(S)$.

Remark 4.3. Elements of the mapping class group are called *mapping classes.*

Example 4.1. From Proposition 2.1, id_{S^1} and f_θ belong to the same mapping class in $\mathrm{MCG}(S^1)$. In fact, every homeomorphism of S^1 is homotopic to id_{S^1}, so $\mathrm{MCG}(S^1)$ is trivial.

Example 4.2. Similarly, from Proposition 2.2, id_{S_2} and f_θ belong to the same mapping class in $\mathrm{MCG}(S^2)$.

Theorem 4.1 (Alexander Lemma)**.** The mapping class group of the closed disk D^2 is trivial.

Proof. Let $f : D^2 \to D^2$ be a homeomorphism and assume that $f(x) = x$ for any $x \in \partial D^2$. We want to show that f is homotopic to id_{D^2}. Consider the map $H : D^2 \times [0,1] \to D^2$, where

$$H(x,t) = \begin{cases} x, & \text{if } 1-t \leq |x| \leq 1 \\ (1-t)f(\frac{x}{1-t}), & \text{if } 0 \leq |x| < 1-t \end{cases}$$

for $t \in [0,1)$. Moreover, define $H(x,1) = id_{D^2}$.
Note that H is continuous, and thus, the homotopy between f and id_{D^2}. Therefore, $\mathrm{MCG}(D^2) = \{id_{D_2}\}$. $\square$

Remark 4.4. We call the previous proof, the *Alexander trick.*

Now we explore mapping class groups of infinite order, such as that of the annulus and torus.

4.2 The simplest infinite order mapping class group

Definition 4.3 (path)**.** Let X be a topological space. A path in X is a continuous function $f : [0,1] \to X$.

Definition 4.4 (simply-connected)**.** A topological space X is simply-connected if any loop in the space can be continuously deformed into a single point, i.e. is *contractible.*

Remark 4.5. The fundamental group of X at each point in the space measures how far X is from simply-connectedness. A *path-connected* space is simply-connected if and only if its fundamental group is trivial.

Remark 4.6. A surface S is simply-connected if and only if it is *connected* with genus 0.

Example 4.3. It follows from Remark 4.6 that S^2 is simply-connected.

Definition 4.5 (covering)**.** A covering of a space X is another space E together with a map $\Phi : E \to X$ such that for any point $x \in X$, there exists an open neighborhood U of x such that $\Phi^{-1}(U)$ is a disjoint union of open sets in E, each of which is mapped homeomorphically onto U.

Definition 4.6 (Universal cover)**.** A universal cover of X is a covering space that is simply-connected.

Remark 4.7. If a connected topological space X is simply-connected, then it is its own universal cover.

Example 4.4. By Example 4.3 and Remark 4.7, S^2 is its own universal cover.

Example 4.5. $\mathbb{R}$ is the universal cover of S^1. Note that $\mathbb{R}$ is a simply-connected space with the covering map $f : \mathbb{R} \to S^1$ such that $f(t) = e^{2\pi i t}$.

Example 4.6. Let A be an annulus. The universal cover of A is the infinite strip $\tilde{A} \approx \mathbb{R} \times [0, 1]$ since A is homeomorphic to $S^1 \times [0, 1]$.

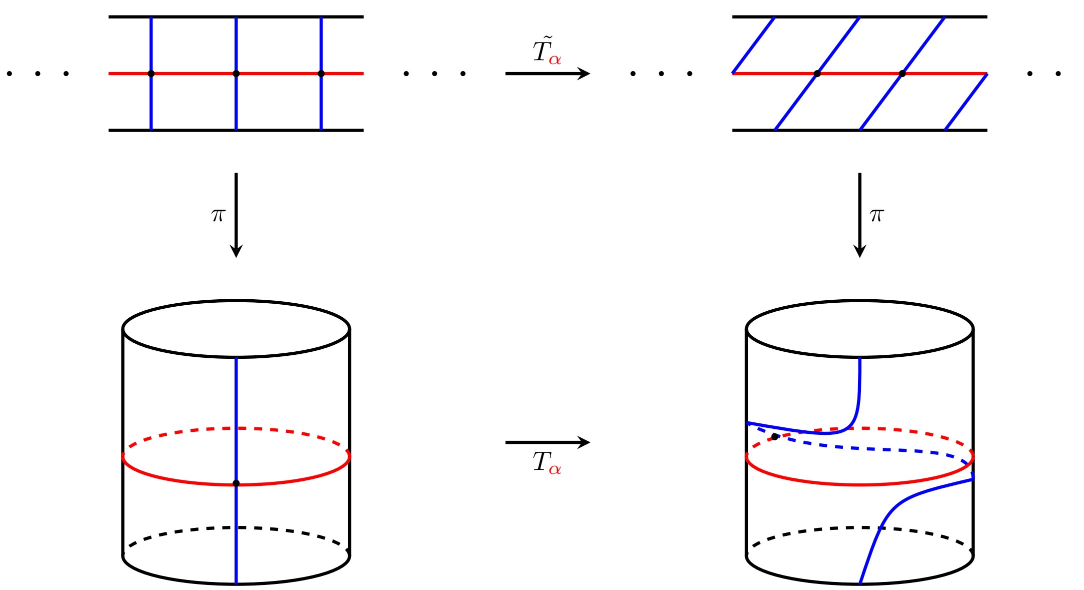

Figure 7: A preferred lift of a Dehn twist about the red simple closed curve α on A.

Theorem 4.2. $\mathrm{MCG}(A) \approx \mathbb{Z}$.

Proof. We construct a map $\rho : \mathrm{MCG}(A) \to \mathbb{Z}$. Let $f \in \mathrm{MCG}(A)$ and let $\varphi : A \to A$ be any homeomorphism representing the mapping class f. Then φ has a preferred lift $\tilde{\varphi} : \tilde{A} \to \tilde{A}$ such that $\tilde{\varphi}|_{\mathbb{R}\times\{0\}} = id|_{\mathbb{R}\times\{0\}}$. Now let $\tilde{\varphi_1} : \mathbb{R} \to \mathbb{R}$ denote the restriction $\tilde{\varphi}|_{\mathbb{R}\times\{1\}}$. Note that we can canonically identify $\tilde{\varphi_1}$ with $\mathbb{R}$. Next, we define $\rho(f) = \tilde{\varphi_1}(0)$. Notice that any homeomorphism homotopic to identity satisfies $\tilde{\varphi_1}(0) = 0$ since $\tilde{\varphi}|_{\mathbb{R}\times\{1\}} = id|_{\mathbb{R}\times\{1\}}$. For any homeomorphism that is not homotopic to identity, $\tilde{\varphi_1}(0) = n$, where $n \in \mathbb{Z}\backslash\{0\}$. This follows from the fact that ∂A is fixed pointwise and we only consider simple closed curves on A, so there cannot be any intersections of arcs in $\tilde{A}$. Hence, $\rho(f) = \tilde{\varphi_1}(0) \in \mathbb{Z}$. Since compositions of homeomorphisms of A map to compositions of integer translations of $\mathbb{R}$, it is clear that ρ is a homomorphism.

Further, we find that $\ker(\rho)$ is trivial since ∂A is fixed pointwise and any homeomorphism homotopic to identity satisfies $\tilde{\varphi_1}(0) = 0$. Every homeomorphism homotopic to identity lifts to arcs homotopic to the ones shown in Figure 7 on the left infinite strip. Thus, ρ is injective. And surjectivity follows from the existence of a homeomorphism for each integer translation. See [2] for a more detailed proof. □

Remark 4.8. Every homeomorphism that is homotopic to T_α maps to $1 \in \mathbb{Z}$ under the map ρ. The integer translation for the mapping class of T_α is depicted in Figure 7.

4.3 The mapping class group of the torus

Theorem 4.3. $\mathrm{MCG}(T^2) \cong \mathrm{SL}_2(\mathbb{Z})$.

Proof. (Sketch) We use a similar method as in Theorem 4.2. Construct a map $\sigma : \mathrm{MCG}(T^2) \to \mathrm{SL}_2(\mathbb{Z})$. Note that $\mathbb{R}^2$ is the universal cover of T^2. Let T_α be a Dehn twist about α on T^2, which can clearly be representative of an element of $\mathrm{MCG}(T^2)$. Then T_α has a preferred lift $\tilde{T_\alpha} : \mathbb{R}^2 \to \mathbb{R}^2$. See Figure 8.

Every simple closed curve on a torus can be homotoped to intersect a point and lifts to a line through the origin which also passes through another integer point. In fact, the first such point is (n, m), where $\gcd(n, m) = 1$. Moreover, since there is a bijective correspondence between nontrivial homotopy classes of oriented simple closed curves on T^2 and the primitive elements of Z^2, there must exist some matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ such that $A((n, m)) = (1, 0)$. Notice that $\tilde{T_\alpha}(1, 0) = (1, 0)$ and $\tilde{T_\alpha}(0, 1) = (1, 1)$. Thus, $\tilde{T_\alpha}$ is a linear, orientation-preserving homeomorphism of $\mathbb{R}^2$ preserving $\mathbb{Z}^2$. It follows that $\tilde{T_\alpha}$ is isomorphic to T_α, where $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T_\alpha(1, 0) = (1, 0)$ and $T_\alpha(0, 1) = (1, 1)$. So we can represent a Dehn twist about α as

$$T_\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Now let T_β be a Dehn twist about β, which is another representative of a mapping class in $\mathrm{MCG}(T^2)$. Then we deduce that $\tilde{T_\beta}$ is isomorphic to T_β, where $T_\beta : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

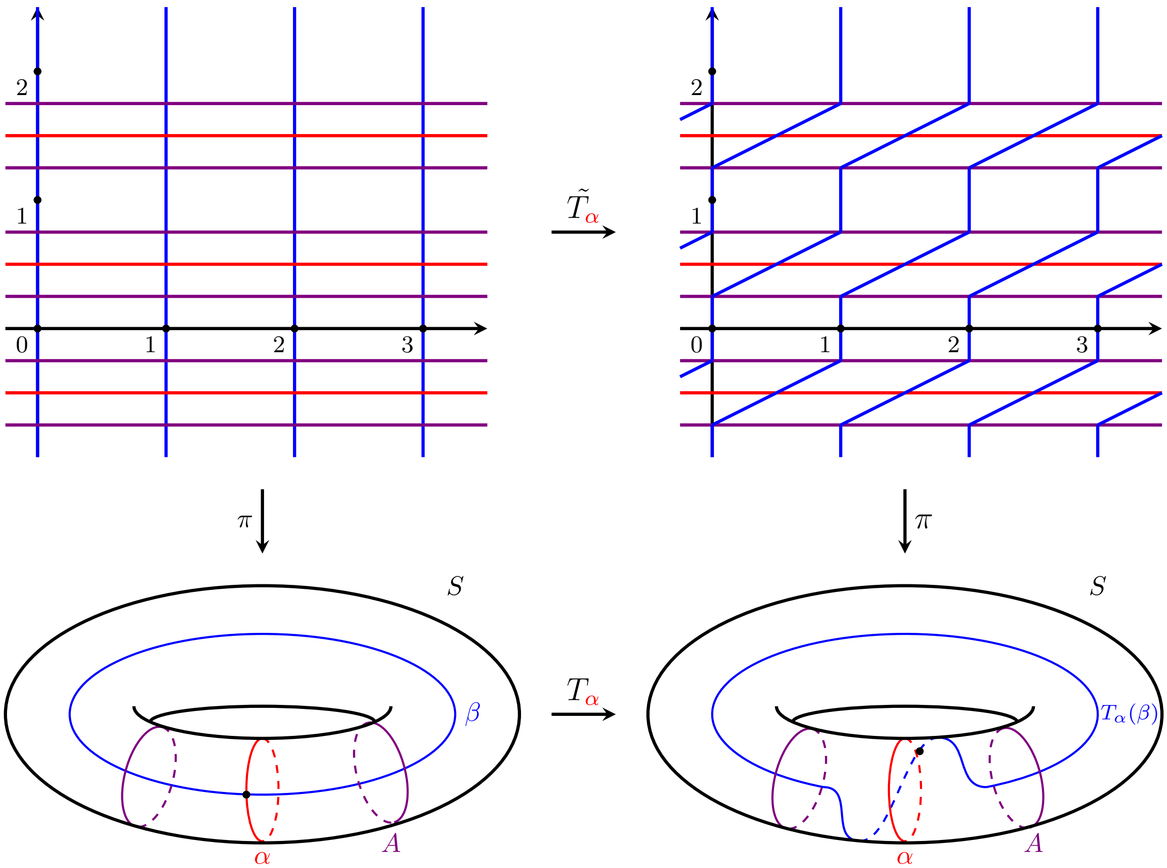

Figure 8: A preferred lift of a Dehn twist about α on T^2.

$T_\beta(1,0) = (1,-1)$ and $T_\beta(0,1) = (0,1)$. Hence, we can represent a Dehn twist about β as

$$T_\beta = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Recall that $\mathrm{SL}_2(\mathbb{Z})$ is the set of all 2×2 matrices with integer entries and determinant 1. Moreover, it is generated by the matrices T_α and T_β. It turns out that the mapping class group of the torus is generated by the same matrices. See [2] for the full proof. $\square$

Theorem 4.4. The mapping class group of a compact orientable surface is generated by Dehn twists.

4.4 Connection to braid groups

The notion of a mathematical braid is classical with a historical record that dates back to thousands of years ago. They entered the realm of mathematics a couple of centuries ago and one of their first appearance was in Gauss's study of knots in the early 19th century and Hurwitz's paper on Riemann surfaces in 1891. Soon after, in 1925, Artin formally defined a braid group and now there is a modern mathematical study of braids that offers new perspectives of these enchanting objects.

Informally, we define a *braid* to be an intertwining of n strings, where $n \in \mathbb{N}$ and the strings remain at separate ends, i.e. they do not fuse together. We use σ_i to describe the braid in which the i-string crosses over the $i+1$-string. Each n-braid can be expressed as a *braid word* (e.g $\sigma_1\sigma_2\sigma_1^{-1}$ in B_3).

Definition 4.7 (Braid group)**.** The braid group on n strands, denoted B_n, is the group of equivalence classes of $n-$braids.

Remark 4.9. The group operation for B_n is concatenation, which is clearly associative, the identity braid does not contain any strings that cross over another, and the inverse of a braid "undoes" the original braid.

Theorem 4.5. The braid group B_n has the presentation

$$B_n = \langle \sigma_1, \sigma_2, \cdots, \sigma_{n-1} \,|\, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1 \text{ and } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq n-2\rangle.$$

Proposition 4.1. $B_2 \cong \mathbb{Z}$.

Proof. By Theorem 4.5, the braid group B_2 has the presentation

$$B_2 = \langle \sigma_1 \rangle.$$

Since B_2 is generated by a single element, and thus cyclic, it follows that B_2 is isomorphic to $\mathbb{Z}$. □

Remark 4.10. $B_2 \approx \mathrm{MCG}(A)$.

Proposition 4.2. $B_3 \cong \langle x, y \,|\, x^2 = y^3 \rangle$.

Proof. Let $G = \langle x, y \,|\, x^2 = y^3\rangle$. Then we can define the map

$$\begin{aligned} f : G &\to B_3 \\ x &\mapsto \sigma_1\sigma_2 \\ y &\mapsto \sigma_1\sigma_2\sigma_1. \end{aligned}$$

We check that f is well-defined. By Theorem 4.5, we have that $B_3 = \langle \sigma_1, \sigma_2 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\rangle$, so

$$f(x^3) = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1 = f(y^2).$$

Thus, $x^3 = y^2$ implies that $f(x^3) = f(y^2)$. Now it remains to find f^{-1} and check that it is also well-defined. Consider the map

$$\begin{aligned} f^{-1} : B_3 &\to G \\ \sigma_1 &\mapsto x^2y^{-1} \\ \sigma_2 &\mapsto yx^{-1}. \end{aligned}$$

Since $f \circ f^{-1} = 1_{B_3}$ and $f^{-1} \circ f = 1_G$, we deduce that B_3 is ismomorphic to G. Therefore, we have shown $B_3 \cong \langle x, y \,|\, x^2 = y^3 \rangle$. □

Definition 4.8 (Configuration space)**.** A configuration space is the set of all possible *ordered* configurations of n particles

$$C_n(\mathbb{R}^2) = \{(p_1, p_2, \cdots, p_n) \in (\mathbb{R}^2)^n \ \ | \, p_i \neq p_j \text{ for } i \neq j\},$$

where $p_i \neq p_j$ is the condition that the particles must not collide. We can also consider the set of all possible *unordered* configurations of n particles

$$UC_n(\mathbb{R}^2) = \{\{p_1, p_2, \cdots, p_n\} \subset \mathbb{R}^2 \ \ | \, p_i \neq p_j \text{ for } i \neq j\}.$$

Remark 4.11. Note that the left wall and right wall of a configuration space represent points in $UC_n(\mathbb{R}^2)$. In fact, they can be realized as the same point so we can form the notion of a *loop* in the space.

Theorem 4.6. The fundamental group of $UC_n(\mathbb{R}^2)$ is isomorphic to the braid group B_n. See [1] or [2] for more details.

Now we return to mapping class groups and consider a disk with n punctures D_n. Then define a map $\varphi : B_n \to \mathrm{MCG}(D_n)$. Given a braid, we slide the disk across the braid to obtain a homeomorphism. We can visualize this as follows: each puncture is connected by a string to the boundary of the disk and each mapping homomorphism that permutes two of the punctures can then be seen to be a homotopy of the strings, i.e. a braid. It turns out that φ is indeed an isomorphism.

Remark 4.12. By $\mathrm{MCG}(D_n)$, we denote the group of mapping classes of homeomorphisms of an n-punctured disk which fix points on the boundary of the circle pointwise, but not necessarily the n punctures.

Theorem 4.7. The mapping class group of an $n-$punctured disk $\mathrm{MCG}(D_n)$ is isomorphic to the braid group B_n.

Remark 4.13. Using this mapping class group interpretation of braids, each braid can be classified as *periodic, reducible* or *pseudo-Anosov* (Nielsen–Thurston classification).

In this subsection, we have been exposed to three distinct perspectives of braid groups, namely, as traditional braids, configuration spaces and punctured disks. Remarkably, the bridge between braid groups and mapping class groups serves as a guidepost for the Birman–Hilden theory. This connection to braid groups illuminates the relevance of mapping class groups, with numerous additional linkages to indulge.

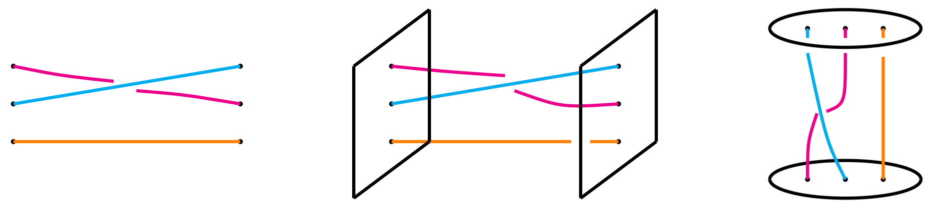

Figure 9: Three perspectives of B_3: traditional braid, configuration space and 3-punctured disk.

Summary

We have surveryed the basics of mapping class groups. Significant players are homeomorphisms, especially Dehn twists, and their mapping classes. As generators of the mapping class group, a profound understanding of these functions provides insight on the surfaces in consideration. In particular, we have coined the term 'symmetries' to describe a certain discrete group (i.e. a topological group that does not contain any limit points) that is associated with the space. Mapping class groups frequent many areas of mathematics and their connection to braid groups exemplifies this phenomenon.

This prompts some ideas for future areas of research.

- Which groups are also mapping class groups?
- What are other generating sets for the mapping class group?
- How can we use representations of braid groups to deduce that the mapping class group of a genus 2 surface is a linear group?

References

[1] Matt Clay and Dan Margalit. *Office Hours with a Geometric Group Theorist.* Princeton University Press, Princeton, NJ, 2017.

[2] Benson Farb and Dan Margalit. *A Primer on Mapping Class Groups.* Princeton University Press, Princeton, NJ, 2012.

[3] James R. Munkres. *Topology.* Prentice Hall, Inc., Upper Saddle River, NJ, third edition, 2000.